Penny Auctions*

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Abstract

This paper studies penny auctions, a novel auction format in which every bid increases the price by a small amount, but placing a bid is costly. Outcomes of real-life penny auctions are often surprising. Even when selling cash, the seller may obtain revenue that is much higher or lower than its nominal value, and losers in an auction sometimes pay much more than the winner. This paper characterizes all symmetric Markov-perfect equilibria of penny auctions and studies penny auctions' properties. The results show that a high variance of outcomes is a natural property of the penny auction format and high revenues are inconsistent with rational risk-neutral participants.

JEL: D11, D44, C73

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1 Introduction

A typical penny auction sells a new brand-name gadget at a starting price of zero dollars, and sets a timer at one minute. When the auction starts, the timer starts to tick down, and players may submit bids. Each bid costs one dollar, increases the price by one cent, and resets the timer to one minute. Once the timer reaches zero, the last bidder is declared the winner and can purchase the object at the final price. Penny auctions are similar to

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English auctions, but with one significant difference: in penny auctions, bidders have to pay bid fees for each price increment.

Penny auctions have surprising properties in practice. (For a detailed analysis and description, see Appendix A.) First, the relation between the final price and the value of the object is stochastic—similar objects are often sold at very high final prices and often very low final prices. Second, the winner of the auction usually pays less than the value of the object. However, because the losers collectively pay large amounts, the revenue to the seller is often higher than the value of the object. In fact, the losers sometimes pay more than the winner.

Penny auctions are interesting for several reasons. First, penny auctions are popular in real life, and they are not a special case of any well-known auction format, so it is important to know their properties. In this paper, I characterize all symmetric Markov-perfect equilibria in penny auctions and show that, even under standard assumptions (rational risk-neutral bidders; common value of the prize; and common knowledge of the number of players), equilibria have most of the characteristics described above. Second, the popularity of penny auctions has caught the attention of policymakers who are asking whether they might be scams or games of chance.[1] In this paper, I show that, although penny auctions do not use any randomization devices, equilibria in penny auctions involve mixed strategies. Thus, from the individual bidder's perspective, the penny auction format is similar to that of a lottery. Third, I argue that high revenues in penny auctions cannot be explained by rational behavior of risk-neutral agents; therefore, penny auctions may provide valuable empirical evidence for testing theories in behavioral economics.

The goal of this paper is to study properties of penny auctions under standard assumptions. The main contribution of the paper is the characterization of *all* symmetric Markov-perfect equilibria in penny auctions with rational risk-neutral participants. This allows me to answer two questions. First, what are the general properties of penny auctions? More specifically, the aim is to understand which outcomes of penny auctions are due to the auction format and which ones depend on specific assumptions about the parameters of the model or equilibrium selection. Second, how do the specific assumptions made in the penny auctions literature affect the results? Most of the literature on penny actions builds on Augenblick (2016) and Platt, Price, and Tappen (2013) and studies a special case of the model analyzed here. As the results will show, this special case is not

[1] The Better Business Bureau called penny auctions a scam and said that "BBB recommends you treat them the same way you would legal gambling in a casino—know exactly how the bidding works, set a limit for yourself, and be prepared to walk away before you go over that limit" `http://www.bbb.org/dallas/migration/bbb-news-releases/2012/01/bbb-names-top-ten-scams-for-2011/`. On the other hand, the Gambling Commission in the UK said that it "was not convinced that penny auctions amounted to gambling." `http://news.bbc.co.uk/2/hi/business/7793054.stm` (Both accessed on April 4, 2015)

without a loss of generality, and is not robust in small variations in the assumptions.

This paper is one of the first to study penny auctions with two others being Augenblick (2016) and Platt, Price, and Tappen (2013).[2] In this paper, I show that penny auctions with rational, risk-neutral agents cannot explain the seller's high average profit. This fact has inspired subsequent literature to extend the model with more complex preferences. Because the auction format leads to highly uncertain outcomes, risk-loving individuals would be happy to pay more than the expected value of winning. Platt, Price, and Tappen (2013) finds some evidence for this. Gnutzmann (2014) extends the analysis to cumulative-prospect-theory preferences and argues that it explains the behavior even better than standard risk-loving behavior.[3] Augenblick (2016) proposes that the explanation could be another behavioral bias—the sunk-cost bias. Caldara (2012) used lab experiments and found that high revenues came mainly from the agents who are inexperienced and not strategically sophisticated.

There are forms of pay-to-bid auctions other than penny auctions. As in penny auctions, most revenue comes from bid fees rather than the winning price. A *price-reveal auction* is a descending-price auction in which the current price is hidden and bidders can privately observe the price for a fee. Gallice (2016) shows that under the standard assumptions those auctions would end quickly and would be unprofitable. In *unique-price auctions*, bidders submit positive integers as bids, and the winner is the one who submitted the lowest or highest unique number. Raviv and Virag (2009) and Östling, Wang, Chou, and Camerer (2011) have found a surprising degree of convergence toward the equilibrium in these auctions.

The concept of penny auctions is similar to that of the *dollar auction* introduced in Shubik (1971). In this kind of auction, the auctioneer sells cash to the highest bidder, but the two highest bidders pay their bids. Shubik uses dollar auctions to illustrate potential weaknesses of traditional solution concepts and describes this kind of auction as extremely simple, highly amusing, and usually highly profitable for the auctioneer. The dollar auction is a kind of *all-pay auction* that has been used to model rent seeking, research and development races, political contests, and job promotions. Baye, Kovenock, and de Vries (1996) provides full characterization of equilibria under full information in one-shot (first-price), all-pay auctions. The second-price, all-pay auction (also called a "war of attrition") has been used to study evolutionary stability of conflicts, price wars, bargaining, and patent competition. Full characterization of equilibria under full information is given by Hendricks, Weiss, and Wilson (1988). Although a penny auction

[2] The first versions of all three papers were written independently in 2009.

[3] Hinnosaar (2015) shows that the optimal auctions with risk-loving or prospect theory bidders would ensure very large revenues.

is an all-pay auction, it is not a special case of previously documented auction formats because, in contrast to standard all-pay auctions, the winner of a penny auction might pay less than the losers.

This paper is organized as follows. Section 2 illustrates the game structure and the set of equilibria by a simple example. Section 3 introduces the general model and the equilibrium concept. Section 4 analyzes the case when the price increment is zero and, therefore, the auction game is infinite. Section 5 discusses the case when the price increment is strictly positive. Section 6 contains concluding remarks. There are three appendices: appendix A provides some stylized facts about penny auctions that motivate the paper, appendix B provides proofs of all results, and appendix C gives an example of a penny auction with a positive bid increment that has multiple equilibria.

2 Illustrative Example

Before studying the general model, let us consider a simple example with three participants and simple parametric values. Let the object have value 3 for all participants. The game is discrete, the initial price is 0, and nobody is yet the *leader*. In each period, all *non-leaders* (bidders other than the leader) simultaneously choose whether to bid or not. The current leader cannot bid. Each bid raises the price by 1 and costs 1 to the bidder. One of the participants who chooses to bid becomes the next leader. If at any period there are no bids, the game ends and the current leader will receive the object and pay the current price. If there are no bids in the initial round, the seller keeps the object.

I need to specify what happens when there is more than one simultaneous bid. In this paper I consider two alternative assumptions.

1. First, the *single-bid* case, where only one of the simultaneous bids is randomly accepted. Therefore the price is raised by 1, each of the K bidders submitting a bid have probability $1/K$ of becoming the new leader, and only the new leader pays the bid cost 1.

2. Second, the *simultaneous-bids* case, where all simultaneous bids are accepted. Therefore the price is raised by K, each of the K bidders submitting a bid has again probability $1/K$ of becoming the leader, but all K bidders pay a bid cost of 1.

We are looking for symmetric Markov-perfect equilibria, which as we will see in the general analysis, can be found by backward induction starting from high prices (above 3) and assuming symmetry at every price level.

2.1 The Single-bid Case

We will start with the single-bid case. First note that at prices $p \geq 2$, the non-leaders never want to bid, because submitting a bid would cost 1 and raise the price to at least 3. Therefore once the price reaches 2, the current leader is certain to obtain the object and thus receives a payoff of 1.

Now consider the non-leader's options at price 1. By not submitting a bid, the participant ensures a payoff of 0, because the game will either end at this period or at the next period, but in either case she will not pay and she will not receive anything. If she submits a bid, she will either pay the bid cost 1 and receive a payoff of 1 in the next period (which means total payoff of 0) or her bid is not accepted and she receives 0 as well. Therefore she would be indifferent between submitting a bid or not. This means that any probability $q(1)$ can be part of an equilibrium strategy at price 1. The expected payoff for the current leader is $2[1-q(1)]^2$, because with probability $[1-q(1)]^2$ none of the non-leaders submits a bid, and thus the leader purchases the object at price 1. Let us define a specific probability $\bar{q}$, which makes the leader's value at the price exactly equal to the bid cost 1, i.e. $\bar{q} = 1 - \sqrt{1/2} \approx 0.2929$.

Finally, consider the initial price 0, i.e. the beginning of the game. As discussed above, the continuation value for a non-leader is 0, so by not submitting a bid the players would receive an expected payoff of 0. By submitting a bid, they would receive either $2[1-q(1)]^2$ minus the bid cost 1, or 0, depending on whether the bid is accepted or not. Therefore, depending on $q(1)$, we have three cases:

1. If $q(1) > \bar{q}$, then the leader's value is at a price 1 less than the bid cost, so at price 0 nobody wants to bid. In all these equilibria, the seller keeps the object.

2. If $q(1) < \bar{q}$, then the leader's value is at a price 1 higher than the bid cost, so that at price 0 each participant bids with certainty. In all these equilibria the object is sold with certainty, but expected revenue is less[4] than 3, and the final price is either 1 or 2.

3. If $q(1) = \bar{q}$, then at $p = 0$ all bidders are indifferent between bidding and not bidding, thus any $q(0)$ can be part of an equilibrium strategy. In all those equilibria, the seller may keep the object with some probability, but conditional on a sale, the expected revenue is 3.

As we have seen, there is a continuum of equilibria under these assumptions. Let me highlight two particular equilibria that are of particular interest. I will call $q(1) = \bar{q}$ and

[4]The expected revenue is $1 + [1-q(1)]^2 + \left(1 - [1-q(1)]^2\right)[1+2] < 3$ as $q(1) < 1 - \sqrt{1/2}$.

$q(0) = 1$ the *indifference equilibrium*. This is the equilibrium that is used in essentially all other penny auction papers, starting from Augenblick (2016) and Platt, Price, and Tappen (2013). It is a particularly convenient equilibrium; the participants are always indifferent between bidding and not bidding, the object is sold with certainty, and the revenue is 3. Due to these properties, the equilibrium is convenient to characterize analytically and thus can be used for extensions and empirical analysis.

Secondly, I will call $q(1) = 1$ and $q(0) = 0$ the *robust equilibrium*. I call it this because it is the only equilibrium that is robust to small changes in parameter values. To see this, consider a small deviation in parameter values. For example, let the object be $3 + \gamma$ for some $0 < \gamma < 1$. The analysis for prices above 1 will be unchanged, but at price 1 it is now strictly better to submit a bid rather than not to bid. Given this, it is strictly better to be a non-leader at price 1 than the leader. This means that at price 0 all participants prefer not to bid. Note that this is the unique equilibrium for any $\gamma > 0$.

2.2 The Simultaneous-bids Case

Consider now the simultaneous-bids case, where paying the bid cost does not guarantee that the bidder becomes the new leader. Again, at any price $p \geq 2$ the game ends instantly. But now at price 1, the trade-off is slightly different than with the single-bid case. Suppose that in equilibrium bidders submit bids with positive probability. Then submitting a bid costs 1 with certainty, but ensures the leader position (and thus future payoff of 1) with probability that is strictly less than 1, because the other non-leader also submits the bid with positive probability. In this case a random bidder would be chosen as the next leader. Thus, in equilibrium we must have that $q(1) = 0$.

Now, consider the initial price 0. Then the only symmetric equilibrium is such that all bidders bid with probability $0 < q(0) < 1$ and the expected payoff from submitting a bid is equal to zero. This implies $q(0) = (3 - \sqrt{5})/2 \approx 0.3820$. In equilibrium, the support of final prices is $\{0, 1, 2, 3\}$ and conditional on sale, the expected revenue is 3.

Finally, this unique equilibrium is also robust to changes in parameters. For example, if the value of the object is $3 + \gamma$ for small $\gamma > 0$, then $q(1) = \gamma > 0$ and $q(0)$ is slightly changed, but as $\gamma \to 0$, the equilibrium would converge to the equilibrium characterized above.

2.3 Zero Price Increment

In practice, in some penny auctions the winner of the auction pays a fixed price, which could be either zero or positive. This changes the structure of the game, because as time passes the price does not rise, so there is no point where the game has to end. This means that the game is potentially infinite. Consider the example again, but now suppose that the winner pays 0 regardless of the number of bids submitted so far. Now, there are just two different situations that could arise. First, in the initial period there could be three non-leaders who simultaneously decide whether to bid or not. In all the following periods, assuming that the game has not ended, there are only two non-leaders. In both the single-bid case and the simultaneous-bids case, there is a unique probability with which the two non-leaders bid.[5]

Now, consider the initial period in which all three participants are still non-leaders. In the simultaneous-bids case we can determine the equilibrium probability in the same way[6]—by equating the expected gains from submitting a bid with the bid cost 1.

In the single-bid case, however, the continuation value from submitting a bid is exactly equal to the bid cost regardless of the number of non-leaders. Therefore the probability with which the participants bid in the initial period is not determined by the equilibrium conditions and any probability could be supported by an equilibrium. I will call the equilibrium where initial bidding probability is 1 the "indifference equilibrium".

2.4 General Comparison

The properties highlighted in the previous example hold more generally. Let's first summarize the results under the assumption of "generic" parameters, meaning that the difference between the value of the object and the bid cost is not exactly divisible by the bid fee. In this case the results are summarized by Table 1. In three out of four assumptions the final price reaches very high values with positive probability and the expected revenue is at most the value of the object. These properties together imply that the outcome of the auction (measured either in final price or realized profits) is stochastic, because both very high and very low values can be reached with positive probability.

However, the fourth assumption—a positive price increment combined with the single-bid case—differs from this general picture. In this final case, the auction either does not start or it ends at a very low price. As this result is not supported by empirical evidence, the literature has focused on a non-generic parametrization, where the difference between the value of the object and the bid cost is exactly a multiple of the bid increment.[7] In this case the results change and the set of equilibria is much richer, including the "indifference equilibrium". The results under this assumption are summarized by Table 2.

[5]In the single-bid case the probability is higher (≈ 0.4226) than in the simultaneous-bids case (≈ 0.3620), because the effective cost of submitting a bid is lower in the former case than in the latter.

[6]It is slightly lower (≈ 0.1935), because there are three non-leaders instead of two.

[7]Of course, this comparison can be made only when the bid increment is positive. When the bid increment is zero, the qualitative results are unchanged.

Table 1: Properties of equilibria under four assumptions in the case of generic parameters. Highest p on the support and expected revenue are expressed conditional on sale. “Very low” means a number less than or equal to the bid cost plus the bid increment. “Very high” means at least the value of the object minus a very low number.

	Positive price increment		Zero price increment	
	Single-bid	Simultaneous-bids	Single-bid	Simultaneous-bids
Number of SMPE	one	finite	continuum	one
Probability of sale	0 or 1	≤ 1	any	< 1
Highest p support	very low	very high	very high	very high
Expected revenue	very low	$\leq$ value	value	value

Table 2: Properties of equilibria under four assumptions in the case of non-generic parameters. Highest p on the support and expected revenue are expressed conditional on sale. “Very low” means a number less than or equal to the bid cost plus the bid increment. “Very high” means at least the value of the object minus a very low number.

	Positive price increment		Zero price increment	
	Single-bid	Simultaneous-bids	Single-bid	Simultaneous-bids
Number of SMPE	continuum	finite	continuum	one
Probability of sale	any	≤ 1	any	< 1
Highest p support	any	very high	very high	very high
Expected revenue	$\leq$ value	$\leq$ value	value	value
“Nice equilibrium”				
Probability of sale	1	≤ 1	1	< 1
Highest p support	very high	very high	very high	very high
Expected revenue	value	$\leq$ value	value	value
“Robust equilibrium”				
Probability of sale	0 or 1	≤ 1	any	< 1
Highest p support	very low	very high	very high	very high
Expected revenue	very low	$\leq$ value	value	value

Focusing on the indifference equilibrium, we see that equilibria under all four assumptions have the properties mentioned above: the expected revenue is at most the value of the object, but the outcomes are stochastic. However, under the positive bid increment and the single-bid case, the observation is not robust to small changes to parameter values. The robust equilibria have the same properties as the equilibria under the generic parameters as described above.

3 Model

The auctioneer sells an object with common value V. There are $N + 1 \geq 3$ players (bidders) participating in the auction,[8] denoted by $i \in \{0, 1, \ldots, N\}$. All bidders are risk-neutral and at each point in time maximize the expected difference between the future costs and benefits. The auction is dynamic, and players submit bids at discrete time points $t \in \{0, 1, \ldots\}$. The auction starts at initial price P_0. At time $t = 0$, all $N + 1$ bidders are non-leaders. At each period $t > 0$, exactly one of the players is the current leader, and the other N players are non-leaders.

In each period t, non-leaders simultaneously choose whether to submit a bid or pass. The leader cannot bid.[9] Each submission of a bid costs C dollars and increments the price by ε. If $K > 0$ of the non-leaders submit a bid, each of them has equal probability $1/K$ to be the leader in the next period. In modeling simultaneous bids by multiple agents, let's consider two cases.

First, the simultaneous-bids case: When $K > 0$ players choose to bid, all K are accepted so that the price increases according to $P_{t+1} = P_t + \varepsilon K$, and each of these K players pays C to the seller. Second, the single-bid case: When $K > 0$ players choose to bid, only one bid is accepted so that the price increases according to $P_{t+1} = P_t + \varepsilon$, and only the new leader pays C to the seller. Note that in real penny auctions, players can submit bids in real time, and all bids are accepted. The single-bid case captures one extreme, where the players can instantly respond to opponents' bids, and the simultaneous-bids case captures the situation where it takes time to respond to opponents' actions and therefore nearly simultaneous bids are all accepted.

If all non-leaders pass at time t, the auction ends. If the auction ends at $t = 0$, then the seller keeps the object, and if it ends at $t > 0$, then the current leader may[10] purchase the object at price P_t. Finally, if the game never ends, all bidders receive payoffs of $-\infty$, and the seller keeps the object. All the parameters of the game are commonly known, and the players know the current leader and observe all previous bids by all players.

I use the following normalizations. In case $\varepsilon > 0$, I normalize $v = (V - P_0)/\varepsilon$, $c = C/\varepsilon$, and $p_t = (P_t - P_0)/\varepsilon$. In case $\varepsilon = 0$, I use $v = V - P_0$, $c = C$, and $p_t = P_t - P_0$.[11] Therefore,

[8] When the number of bidders is either one or two, the game differs significantly because there are no simultaneous bids after period zero. When $N + 1 \leq 2$, it is straightforward to verify that the game ends after at most one round of bids in all equilibria.

[9] This is a restrictive assumption that affects the results, as in some situations the leader may have an incentive to bid. However, in real world penny auctions the leader cannot bid, therefore the assumption is reasonable.

[10] Clearly, if the final price $P_t > V$, the winner will not purchase the object.

[11] Note that in the two cases, $\varepsilon > 0$ and $\varepsilon = 0$, the structure of the game is different: the former will be finite and the latter infinite, so we need to handle the two cases separately, and there is no reason to expect similarities in the equilibria in these two settings.

in all cases, $p_0 = 0$. Given the assumption and normalizations, a penny auction is fully characterized by (N, v, c, ε), where ε is only used to distinguish between the zero-price-increment and positive-price-increment cases.

Finally, I assume that $v - c > 1$ because, otherwise, the auction would never start.[12]

3.1 Symmetric Markov-Perfect Equilibria

I focus on symmetric Markov-Perfect equilibria (SMPE), which impose two additional requirements to the standard subgame-perfect Nash equilibria. To define the equilibrium concept formally, we need more notation. Let the vector of bids at period t be denoted by $\mathbf{b}^t = (b_0^t, \ldots, b_N^t)$, where $b_i^t = 1$ if player i submitted a bid at period t and 0 otherwise. Let the leader after period[13] t be $l^t \in \{0, \ldots, N\}$. The information that each player has when making a choice at period t, or history at t, is $\mathbf{h}^t = (\mathbf{b}^0, l^0, \mathbf{b}^1, l^1, \ldots, \mathbf{b}^{t-1}, l^{t-1})$. The game sets some restrictions on the possible histories. In particular, to become a leader, one must submit a bid, therefore $b_{l^t}^t = 1$. The leader cannot submit a bid, so $b_{l^{t-1}}^t = 0$. Also, $\mathbf{h}^t$ is defined only if none of the previous bid vectors $\mathbf{b}^\tau$ is the zero vector. Denote the set of all possible t-stage histories by $\mathcal{H}^t$ and the set of all possible histories by $\mathcal{H} = \bigcup_{t=0}^{\infty} \mathcal{H}^t$.

In this game, a pure strategy of player i is $b_i : \mathcal{H} \to \{0, 1\}$, where $b_i(\mathbf{h}^t) = 1$ means the player submits a bid at $\mathbf{h}^t$ and $b_i(\mathbf{h}^t) = 0$ means that the player passes. The strategies[14] of the players are $\sigma_i : \mathcal{H} \to [0, 1]$ such that $\sigma_i(\mathbf{h}^t)$ is the probability that player i submits a bid at history $\mathbf{h}^t$. Note that by the rules of the game, $\sigma_i^t(\mathbf{h}^t) = 0$ at all histories $\mathbf{h}^t$, where $l^t = i$. (The leader can only pass.)

We will call a strategy profile symmetric if swapping the identities of two players will not change the strategy profile. Intuitively, the symmetry assumption means that the players' identities do not play any role. This assumption fits because players are anonymous in all real penny auctions. The formal definition is the following.

Definition 1. A strategy profile σ is called symmetric if for all $t \in \{0, 1, \ldots\}$, for all $i, \hat{i} \in \{0, \ldots, N\}$, and for all $\mathbf{h}^t = (\mathbf{b}^\tau, l^\tau)_{\tau=0,\ldots,t-1} \in \mathcal{H}^t$; if $\hat{\mathbf{h}}^t = (\hat{\mathbf{b}}^\tau, \hat{l}^\tau)_{\tau=0,\ldots,t-1} \in \mathcal{H}^t$

[12] In the case $\varepsilon = 0$, it suffices to assume that $v - c > 0$.

[13] That is, the leader is the non-leader that submitted a bid at t and became the leader.

[14] The game has perfect recall, so by Kuhn's theorem, any mixed strategy profile can be replaced by an equivalent behavioral strategy profile. For simpler notation, *strategies* in the text means behavioral strategies.

satisfies

$$\hat{b}_j^\tau = \begin{cases} b_j^\tau & \forall j \notin \{i, \hat{i}\}, \\ b_i^\tau & j = \hat{i}, \\ b_{\hat{i}}^\tau & j = i, \end{cases} \qquad \hat{l}^\tau = \begin{cases} l^\tau & l^\tau \notin \{i, \hat{i}\}, \\ i & l^\tau = \hat{i}, \\ \hat{i} & l^\tau = i, \end{cases} \qquad \forall \tau = \{0, \ldots, t-1\}, \tag{1}$$

then $\sigma_i(\hat{\mathbf{h}}^t) = \sigma_{\hat{i}}(\mathbf{h}^t)$.

Let $\mathcal{S}$ be the set of states in the game, and let $S : \mathcal{H} \to \mathcal{S}$ be the function mapping histories to states. More specifically,

(i) If $\varepsilon = 0$, then $\mathcal{S} = \{N+1, N\}$, and $S(\mathbf{h}^t) = N+1$ if $t = 0$ and N otherwise. Because the price does not increase, the only payoff-relevant characteristic for the players is the number of active bidders, which is $N+1$ in the beginning and N at any round after 0.

(ii) If $\varepsilon > 0$, then $\mathcal{S} = \mathbb{Z}_+$, and $S(\mathbf{h}^t) = \sum_{\tau=0}^{t-1} \sum_{i=0}^{N} b_i^\tau$; that is, the total number of bids made so far or, equivalently, the normalized price. Note that it is unnecessary to explicitly consider two cases with two different numbers of players because at $\mathbf{h}^t = \emptyset$ $S(\mathbf{h}^t) = 0$ and at any other history $S(\mathbf{h}^t) > 0$.

We call a strategy profile Markovian if at any two histories that lead to the same state, the strategies prescribe the same probabilities of bidding. Players using Markovian strategies only condition their behavior on the current price and the number of active bidders. They do not condition their behavior on the whole history of bids or on identities of leaders. In real penny auctions, the only information revealed to buyers is the identity of the leader and the current price, so this restriction also matches real-life penny auctions. The formal definition of Markovian strategy profile is the following.

Definition 2. A strategy profile σ is *Markovian* if for all $\mathbf{h}^t = (\mathbf{b}^\tau, l^\tau)_{\tau=0,\ldots,t-1} \in \mathcal{H}$, $\hat{\mathbf{h}}^{\hat{t}} = (\hat{\mathbf{b}}^\tau, \hat{l}^\tau)_{\tau=0,\ldots,\hat{t}-1} \in \mathcal{H}$ such that $L_i(\mathbf{h}^t) = L_i(\hat{\mathbf{h}}^{\hat{t}})$ and $S(\mathbf{h}^t) = S(\hat{\mathbf{h}}^{\hat{t}})$, so $\sigma_i(\mathbf{h}^t) = \sigma_i(\hat{\mathbf{h}}^{\hat{t}})$.

We are looking for Symmetric Markov-perfect Equilibria, which impose that the equilibrium strategy profile must be symmetric and Markovian. Note that this restriction still allows players to deviate unilaterally and possibly to non-Markovian strategies.

Definition 3. A sub-game perfect Nash Equilibrium strategy profile σ is SMPE if σ is Symmetric and Markovian.

Focusing on SMPE reduces the set of equilibria as well as simplifies the notation and construction of equilibria. Typically, constructing equilibria would require using general strategy profiles. As Proposition 1 shows, any SMPE can be represented by a vector of probabilities, which can be found by finding a stage-game Nash equilibrium for each state sequentially.

Proposition 1. *A strategy profile is an SMPE if and only if it can be represented by* $\mathbf{q}$*, where* $q(\cdot)$ *is the Nash equilibrium of the stage-game in a particular state, taking into account the continuation values.*

In the case $\varepsilon > 0$, the set of states will be the set of all possible prices. Therefore, we can use the current price p (independent of time or history) as the current state variable and solve for a symmetric Nash equilibrium in this state, given the continuation values at states that follow each profile of actions. So the equilibrium is fully characterized by a $q : \mathbb{Z}_+ \to [0, 1]$, where $q(p)$ is the probability of submitting a bid that each non-leader independently uses at price p.

In the case $\varepsilon = 0$, there are only two states. In the beginning of the game there are $N + 1$ non-leaders, and in any of the following histories, the number of non-leaders is N. So the equilibrium is characterized by $(\hat{q}_0, \hat{q})$, where $\hat{q}_0$ is the probability that a player submits a bid at period 0 and $\hat{q}$ is the probability that a non-leader submits a bid in any of the following periods. The SMPE can be found simply by solving for Nash equilibria at both states, taking into account the continuation values.

4 Auctions with Zero Price Increment

When the price increment is zero ($\varepsilon = 0$), the auction is similar to an infinitely repeated game. After each round of bids, bid costs are already sunk, and the benefits from winning are the same. By Proposition 1, any SMPE is characterized by a pair $(\hat{q}_0, \hat{q})$, where $\hat{q}_0$ is the probability that a non-leader will submit at round 0, and $\hat{q}$ is the probability that a non-leader submits a bid in any subsequent round. Let $\hat{v}^*$ and $\hat{v}$ be the leader's and non-leaders' continuation values (after round zero), respectively.

Let $\Psi_N(q)$ denote the probability of becoming a new leader, conditional on paying the bid cost c, when there are $N - 1$ other non-leaders who each submit their bids with probability q. In the single-bid case, non-leaders pay the bid cost only when becoming the new leader, so $\Psi_N(q) = 1$. But in the simultaneous-bids case if there are K other bidders who happen to submit a bid, the probability of becoming the leader is $1/(K + 1)$, so that $\Psi_N(q)$ is the expectation of this fraction, given that K is distributed binomially with parameters $(q, N-1)$. This gives us

$$\Psi_N(q) = \sum_{K=0}^{N-1} \binom{N-1}{K} q^K (1-q)^{N-(K+1)} \frac{1}{K+1} = \frac{1-(1-q)^N}{qN}. \tag{2}$$

Proposition 2 shows that with zero price increment, the set of equilibria is analogous to what we found in the example above. In the simultaneous-bids case it is a uniquely defined interior equilibrium, whereas in the single-bid case, the behavior after the first round is unique and interior (determined by the same trade-offs), but in the initial round any bidding probability could be part of equilibrium behavior.

Proposition 2. *When $\varepsilon = 0$, SMPE $(\hat{q}_0, \hat{q})$ is such that $\hat{q} \in (0,1)$ is uniquely determined by the equality $(1-\hat{q})^N \Psi_N(\hat{q}) = c/v$, and*

(i) in the simultaneous-bids case, $\hat{q}_0$ is uniquely defined by $(1-\hat{q})^N \Psi_{N+1}(\hat{q}_0) = c/v$;

(ii) in the single-bid case, there is a continuum of equilibria, one for each $\hat{q}_0 \in [0,1]$.

In the single-bid case, the set of equilibria characterized in Proposition 2 coincides with equilibria in Augenblick (2016) and Platt, Price, and Tappen (2013), but Proposition 2 is more general in two ways. First, Augenblick and Platt, Price, and Tappen prove that these strategy profiles are SMPE, but Proposition 2 additionally shows that there are *no other* SMPE. Second, it also characterizes SMPE for the simultaneous-bids case and thus makes it possible to form conclusions about the robustness of the single-bid assumption.

Corollary 1 illustrates the differences between the equilibria in (fixed-price) penny auctions in the simultaneous-bids case and the single-bid case. First, in the simultaneous-bids case, the equilibrium is unique, and in particular, bidding with certainty in period 0 is not an equilibrium. Second, the probability that at least one player bids in period $t > 0$ is a strictly decreasing function of the number of bidders N. This characteristic arises because the increase in the number of bidders has two effects on the decision to submit a bid: first, for a fixed $\hat{q}$, increasing bidders decreases the probability that the game ends in the next period, and second, increasing bidders decreases the probability of becoming the next leader after submitting a bid. The latter effect is missing in the single-bid case because only the next leader incurs the bid cost. The lack of this second effect simplifies the analysis significantly because the continuation probability is simply equal to c/v. Augenblick (2016) and Platt, Price, and Tappen (2013) cleverly exploit this simplification in their empirical analysis and extensions. Corollary 1, however, emphasizes that their analysis is not robust to changes in the modeling of simultaneous bids.

Corollary 1. *When $\varepsilon = 0$, penny auctions have the following properties:*

(i) *Simultaneous-bids case*

(a) *At any period $t > 0$, the probability that an individual bidder submits a bid is a constant $\hat{q} \in (0,1)$, and the probability that at least one bidder submits a bid is a constant $\hat{Q} = 1 - (1-\hat{q})^N \in (0,1)$. Both $\hat{q}$ and $\hat{Q}$ are strictly decreasing in N and c/v.*

(b) *The probability that an individual bidder submits a bid at period 0 is $\hat{q}_0 \in (0,1)$, and the probability that at least one bidder submits a bid (or equivalently the probability of selling the object) is $\hat{Q}_0 = 1-(1-\hat{q})^N \in (0,1)$. Both the individual probability, $\hat{q}_0$, and the cumulative probability, $\hat{Q}_0$, are strictly decreasing in c/v.*

(ii) *Single-bid case*

(a) *At any period $t > 0$, the probability that an individual bidder submits a bid is a constant $\hat{q} \in (0,1)$, and the probability that at least one bidder submits a bid is a constant $\hat{Q} = 1 - (1-\hat{q})^N \in (0,1)$. The individual probabilities $\hat{q}$ and $\hat{Q}$ are strictly decreasing in N and c/v; the cumulative probability $\hat{Q}$ is strictly decreasing in c/v and independent of N.*

(b) *There is a continuum of equilibria, one for each $\hat{q}_0 \in [0,1]$, where $\hat{q}_0$ is the probability that an individual bidder submits a bid in period 0. The probability that at least one bidder submits a bid (the probability of selling the object), $\hat{Q}_0$, can therefore be any number in $[0,1]$.*

(iii) *The expected value to players is 0.*

(iv) *The expected revenue to the seller, conditional on sale, is v.*

Corollary 2 illustrates that, although all the payoffs are precisely determined in expected terms, any outcome of the game can occur with strictly positive probability in actual realizations. All three implications in the corollary follow from the simple fact that, in equilibrium in each period $t > 0$, the probability that there are more bids is strictly between zero and one.

Corollary 2.

(i) *With positive probability, the seller sells the object after just one bid and receives revenue c. The winner gets $v - c$, and the losers pay nothing.*

(ii) *On the other hand, there is a positive probability that revenue is larger than any fixed number M. Moreover, with positive probability, the revenue is bigger than M, and the winner only pays c.*

5 Auctions with Positive Price Increments

In this section, penny auctions with positive price increments will be analyzed. Following Proposition 1, any SMPE can be characterized by a vector $\mathbf{q} = (q(0), q(1), \dots)$, where $q(p)$ is the non-leaders' probability to bid at price p. It is both necessary and sufficient to check for stage-game Nash equilibria, given the continuation payoffs induced by the chosen actions. For a given equilibrium, I denote the leader's continuation value by $v^*(p)$ and non-leaders' continuation value by $v(p)$.

With a positive price increment, the price increases over time. Therefore, the game must end in finite time—at least when the price rises above the value of the object. Lemma 1 establishes this observation formally and gives an upper bound to the prices at which bidders will still be active.

Lemma 1. *Suppose $\varepsilon > 0$, and fix any SMPE. Then, $q(p) = 0$ for all $p \geq \lfloor v - c \rfloor$*[15]

According to Lemma 1, for any game we have $q(p) = 0, \forall p \geq \lfloor v - c \rfloor$. Define $\gamma = v - c - \lfloor v - c \rfloor$ so that $\gamma \in [0, 1)$. When taking $p = \lfloor v - c \rfloor + K = v - c - \gamma + K$ for $K = 0, 1, \dots$, then $q(p) = 0$, and

$$v^*(p) = \max\{0, v - p\} = \max\{0, c + \gamma - K\}, \quad v(p) = 0. \tag{3}$$

The rest of the game can be considered finite and solved using backward induction. Take $p < \lfloor v - c \rfloor$. If $p > 0$, there are N non-leaders, and if $p = 0$, there are $N + 1$. Denote the number of non-leaders by $\bar{N}$. At each stage, the stage-game equilibrium could be one of three types: (1) all non-leaders submit bids with certainty, (2) all non-leaders choose not to bid, (3) all non-leaders randomize between bidding and not bidding. Next, I show how each of these three situations characterizes $q(p), v^*(p)$, and $v(p)$.

Simultaneous-bids Case

Consider the three situations under the simultaneous-bids assumption. First, there could be a stage-game equilibrium in which all $\bar{N}$ non-leaders submit bids with certainty. In this case, $q(p), v^*(p)$, and $v(p)$ are characterized by the three equalities in conditions C1. This is an equilibrium if none of the non-leaders wants to pass and become a non-leader at price $p + \bar{N} - 1$ with certainty, as characterized by the inequality in conditions C1.

Conditions 1 (C1)**.** $q(p) = 1$, $v^*(p) = v(p + \bar{N})$, and

$$v(p) = \frac{1}{\bar{N}} v^*(p + \bar{N}) + \frac{\bar{N} - 1}{\bar{N}} v(p + \bar{N}) - c \geq v(p + \bar{N} - 1). \tag{4}$$

[15] $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \leq x\}$ is the floor function.

Second, there could be a stage-game equilibrium in which all $\bar{N}$ non-leaders choose to pass. This is characterized in conditions C2.

Conditions 2 (C2)**.** $q(p) = 0$, $v^*(p) = v - p$, and $v(p) = 0 \geq v^*(p+1) - c$.

Finally, there could be a symmetric mixed-strategy stage-game equilibrium in which all $\bar{N}$ non-leaders bid with probability $q \in (0,1)$. This is characterized in conditions C3.

Conditions 3 (C3)**.** $0 < q(p) < 1$,

$$v(p) = \sum_{K=0}^{\bar{N}-1} \binom{\bar{N}-1}{K} q^K (1-q)^{\bar{N}-1-K} \left[\frac{1}{K+1} v^*(p+K+1) + \frac{K}{K+1} v(p+K+1) \right] - c$$

$$= \sum_{K=1}^{\bar{N}-1} \binom{\bar{N}-1}{K} q^K (1-q)^{\bar{N}-1-K} v(p+K), \tag{5}$$

$$v^*(p) = (1-q)^{\bar{N}}(v-p) + \sum_{K=1}^{\bar{N}} \binom{\bar{N}}{K} q^K (1-q)^{\bar{N}-K} v(p+K). \tag{6}$$

Single-bid case: Under the single-bid assumption, the price always rises by 1, so the conditions in C1, C2, and C3 simplify, but the same three cases are still possible. The three sets of conditions take the following forms:

(C1) $q(p) = 1$, $v^*(p) = v(p+1)$, $v(p) = \frac{1}{\bar{N}}[v^*(p+1) - c] + \frac{\bar{N}-1}{\bar{N}} v(p+1)$, and $v^*(p+1) - c \geq v(p+1)$.

(C2) $q(p) = 0$, $v^*(p) = v - p$, $v(p) = 0$, and $v^*(p+1) - c \leq 0$.

(C3) $0 < q(p) < 1$,

$$\begin{aligned} v^*(p) &= (1-q)^{\bar{N}}(v-p) + [1-(1-q)^{\bar{N}}] v(p+1), \\ v(p) &= v(p+1) + \Psi_{\bar{N}}(q)[v^*(p+1) - c - v(p+1)], \\ v^*(p+1) - c &= \left[1 - \frac{(1-q)^{\bar{N}}}{\Psi_{\bar{N}}(q)} \right] v(p+1), \end{aligned} \tag{7}$$

where $\Psi_{\bar{N}}(q) = \frac{1-(1-q)^{\bar{N}}}{q\bar{N}}$. $q \in [0,1]$.

Note that in every equilibrium each $q(p)$ must satisfy either C1, C2, or C3. Therefore, an equilibrium is recursively characterized. However, nothing requires the equilibrium to be unique. In Appendix C's example with $p = 2$, each of the three sets of conditions give different solutions, so there are three different SMPE.

Proposition 3. *When $\varepsilon > 0$, any SMPE $q : \mathbb{N} \to [0,1]$ with corresponding continuation value functions is recursively characterized by C1, C2, or C3 at each $p < \lfloor v - c \rfloor$ and $q(p) = 0$ for all $p \geq \lfloor v - c \rfloor$, where $\lfloor x \rfloor$ is the floor function. An equilibrium always exists but might not be unique.*

Proposition 3 gives a general characterization of all equilibria in penny auctions with a positive bid increment. Although there are often many equilibria, as Corollaries 3 and 4 below show, equilibria in penny auctions with a positive bid increment have some robust features and emphasize the differences between the simultaneous-bids assumption and the single-bid assumption.

First, Corollary 3 shows that under the simultaneous-bids assumption, the outcomes of *all* SMPE are uncertain in the sense that the game may end at a very low price with positive probability as well as at a very high price with positive probability; the result is a very high realized revenue for the seller. However, in all equilibria, the expected revenue is at most v, and in equilibria in which the game may end in each period, the expected revenue is exactly v.

Second, Corollary 4 shows that the set of equilibria under the single-bid assumption follows a different pattern. Namely, in the generic case when $v - c$ is not an integer, the game ends very quickly (at price 0 or 1), and the revenue is very low. In contrast, in the non-generic case when $v - c$ is an integer, there is a large number of equilibria, including the indifference equilibrium which has the characteristics highlighted above—high prices reached with positive probability and expected revenue v.

Corollary 3. *When $\varepsilon > 0$ and the simultaneous-bids assumption holds,* all *SMPE have the following properties:*

1. *Expected revenue conditional on sale $\overline{R} \leq v$. There exist parameter values where in some equilibria $\overline{R} < r$. If $q(p) < 1$ for all p, then $\overline{R} = v$.*

2. *If $\gamma = (v-c) - \lfloor v - c \rfloor > 0$, then $q(\lfloor v - c \rfloor - 1) > 0$. If $\gamma = 0$, then $q(\lfloor v - c \rfloor - 1) > 0$ or $q(\lfloor v - c \rfloor - 2) > 0$ or both. Conditional on sale, the price level $p^* \geq v - c$ is reached with strictly positive probability.*

3. *Conditional on sale, the realized revenue $R \geq v + (c+1)(v - c - 1)$ with strictly positive probability.*

Corollary 4. *When $\varepsilon > 0$ and the single-bid assumption holds, the set of equilibria depends on $\gamma = v - c - \lfloor v - c \rfloor$.*

1. *If $\gamma > 0$, the SMPE is unique. In particular, if $\lfloor v - c \rfloor$ is even, there are no bids, and the seller keeps the object. If $\lfloor v - c \rfloor$ is odd, all bidders submit bids in the first*

period and none afterward, so the revenue is 2*, and the expected value to bidders is* $(v-2)/(N+1)$*.*

2. *If* $\gamma = 0$*, there is a continuum of equilibria including:*

   (a) *The indifference equilibrium in which all players are always indifferent: for all* $p < \lfloor v - c \rfloor$*, the probability is chosen so that* $v(p) = 0 = v^*(p) - c$ *and condition C3 is satisfied. In fact, there is a continuum of such equilibria because* $q(0)$ *could be arbitrary. In this class of equilibria, conditional on sale, each price* $\{1, \dots, \lfloor v - c \rfloor\}$ *occurs with strictly positive probability.*

   (b) *The robust equilibrium which can be found by taking the limit as* $\gamma \to 0$*. In this equilibrium, the game ends either after* 0 *or* 1 *bid depending on whether* $v - c$ *is even or odd.*

6 Discussion

The goal of this paper was to answer two questions. First: What are the general properties of penny auctions with rational risk-neutral buyers? We found two general properties: (1) the outcomes are stochastic with positive probability for both very high and very low final prices, (2) the expected revenue is at most equal to the value of the object sold. These were properties of all SMPE in almost all cases. The only exception was the single-bid positive bid-increment case, where these properties hold under a particular parametrization and equilibrium selection, subsequently termed the indifference equilibrium. Although this may seem specific, this parametrization and equilibrium selection is exactly the one adopted by the rest of the penny auction literature. This brings us to the second question—How do the specific assumptions used in the penny auctions literature affect the results? We saw that although the qualitative properties of the indifference equilibrium are the same as those highlighted above, the equilibrium is not unique and not robust to small changes in the parameter values.

The results of this paper should caution the empirical penny auction researchers that the standard assumptions in the literature are not without loss of generality and the fact that the equilibrium properties are reminiscent of real outcomes of penny auctions are strongly influenced by the specific assumptions. The simultaneous bids assumption seems to be an equally realistic and has much more robust qualitative properties. Therefore, it would be a good idea to use this either as the main specification or at least as a robustness check. The downside of using this assumption is that the equilibrium characterization is not that simple anymore—instead of explicitly defined hazard function one would have to rely on numerically solving the game backwards, but with the processing power available for researchers, this should pose no significant problems.

The model is unable to explain why real penny auctions have average profit margins higher than zero. With rational risk-neutral bidders, the expected revenue from penny auctions cannot be higher than the value of the object regardless of specific assumptions and the equilibrium selection. To explain higher revenues, the model needs to be expanded. Potential explanations include the intrinsic value from participating or winning[16] and risk-loving preferences. It is also possible that the bidders in penny auctions are boundedly rational so that high revenue could be explained by the sunk-cost bias, prospect-theory preferences, or simply mistakes due to inexperience.

Finally, the results from the penny auctions literature point to an interesting area of future research: perhaps, this type of auctions is good for raising money for public goods. Goeree, Maasland, Onderstal, and Turner (2005) showed that it is better to raise money for public goods with all-pay auctions than with winner-pay auctions. Carpenter, Holmes, and Matthews (2014) found experimentally that their physical implementation of penny auctions (which they called "bucket auctions") raised even more money than four alternative all-pay auction formats, and they attributed the difference to sunk-cost sensitivity.

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[16] After all, penny auctions are typically advertised as "entertainment shopping."

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Appendix A Stylized Facts about Penny Auctions

The data used in this appendix comes from `Swoopo.com`, the largest penny auction site at the beginning of May 2009 when data were collected.[17] Data from 61,153 auctions was collected directly from their website; only auctions with complete data were included. Each auction had information about the auction type, the value of the object (suggested retail price), delivery cost, the winner's identity, and the number of free and costly bids the winner made (which were used to calculate "the savings"). The number of observations and some statistics to compare the orders of magnitude are given in Table 3. The main Swoopo auction types that are relevant for this paper are listed below. All four are special cases of the auction format discussed in this paper. In all cases the bid cost is \$0.75.

1. A *penny auction* is an auction with a price increment of \$0.01.

[17] In March 2011, Swoopo went bankrupt, and the website is not accessible anymore.

2. A *regular auction* is an auction with a price increment of \$0.15.

3. In a *fixed-price auction*, the winner pays the pre-announced, fixed price instead of the ending price of the auction.

4. A *free auction* (or *100% off auction*) is a special case of a fixed-price auction in which the winner pays only the delivery charges.

Table 3: General descriptive statistics about the auctions. The symbols v and c refer to normalized variables introduced in Section 3. The average number of bids can be interpreted approximately as the normalized price, p.

Type of auction	Obser-vations	Average value (\$)	Average price (\$)	Norm. value, v	Norm. cost, c	Avg. # of bids
Regular	41760	166.9	46.7	1044	5	242.9
Penny	7355	773.3	25.1	75919.2	75	1098.1
Fixed price	1634	967	64.9	6290.7	5	2007.2
Free	3295	184.5	0	1222	5	558.5
All auctions	61153	267.6	41.4	10236.3	13.4	420.9

Figure 1 describes the distribution of final prices in different auction formats. To compare prices of objects with different values, the plot is normalized by the value of the object. The final price equals the retail price at 100%. Most auction formats have similar distributions with relatively high mass at low values and long tails. It is not surprising that penny auctions are much more concentrated on low values because the bidders have to make 15 times more bids than in other formats to reach any particular price level.

The most intriguing fact in Figure 1 is the positive mass in relatively high prices because the cumulative bid costs to reach high prices can be much higher than the value of the object. The wide range of ending prices implies that the profit margins to the seller and winner's payoff are very volatile. Indeed, Figure 2(a) describes the distribution of the profit margin,[18,19] and there is positive mass in very high profit margins. The figure is truncated at 1000%; there is also positive, but small, mass at much higher margins.

Similarly, Figure 2(b) describes the winner's savings[20,21] from different types of auctions. In this plot, zero on the x-axis means no savings compared to retail price, so to the left of this value, even the winner would have gained by just purchasing the object from a

[18] Profit margin is simply defined as $\frac{\text{End price}+\text{Total bid costs}-\text{Value}}{\text{Value}} \cdot 100$.

[19] In the dataset, 7.12% of the bids were free. I used 92.88% of \$0.75, which is \$0.6966, as the bid cost.

[20] Swoopo.com defines the winner's savings as the difference between the value of the object and winner's total cost divided by the value. Note that the reported savings at the website are such that the negative numbers are replaced by 0.

[21] For the winners, the number of free bids is known, so this is taken into account precisely.

retail store. In most cases, the winner's savings are highly positive, and winner's savings probably explain why agents participate in the auctions after all. The density of the winnings is increasing in all auction formats with the mode near 100%, but the auction formats differ. Regular auctions have a relatively low mean and the flattest distribution, whereas free auctions have highest mean and more mass concentrated near 100%. These results fit expectations because, in these auctions, the cost is relatively more equally distributed between the bidders. (If the winners were making most of the bids, they would win very early.)

Finally, Figure 3 shows the distribution of the total number of bids. The frequencies slowly decrease as the number of bids increases, except in low numbers of bids where the decrease is rapid. In penny auctions there are, on average, many more bids than in other formats.

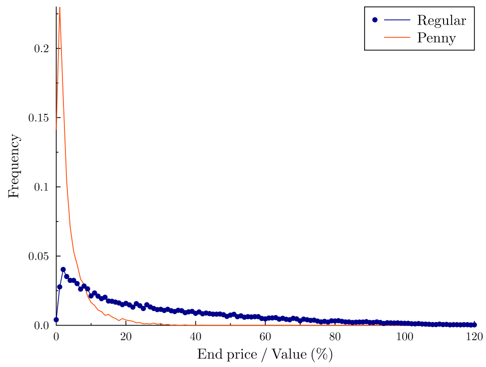
Appendix B Proofs

B.1 Proof of Proposition 1

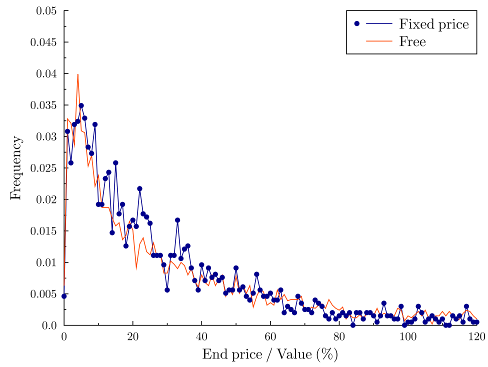
The proof of Proposition 1 consists of three lemmas. First, Lemma 2 proves the representation part; that is, any SMPE can be represented by $q(\cdot)$, and vice versa, if a subgame-perfect Nash equilibrium (SPNE) can be represented by $q(\cdot)$, then it is an SMPE. Lemmas 3 and 4 prove the characterization results for $\varepsilon > 0$ and $\varepsilon = 0$, respectively. In both cases, I show that the game can be essentially solved by looking at one-shot deviations. Note that because the game is without discounting and in both cases (potentially) infinite, continuity at infinity is not satisfied, so it is not valid to simply apply the one-shot deviation principle. The results will prove that it is still possible to use standard characterization techniques.

Lemma 2. *A strategy profile σ is symmetric and Markovian if and only if it can be represented by $q : \mathcal{S} \to [0, 1]$, where $q(s)$ is the probability bidder i bids at state $s \in \mathcal{S}$ for each non-leader $i \in \{0, \dots, N\}$.*

Proof Because q is only defined on states $\mathcal{S}$ and equally for all non-leaders, it is obvious that it is symmetric and Markovian; therefore, sufficiency is trivially satisfied. For necessity, take any symmetric Markovian strategy profile $\sigma = (\sigma_0, \dots, \sigma_N)$. Construct functions $q_0, \dots, q_N$, where $q_i : \mathcal{S} \to [0, 1]$, by setting $q_i(S(h^t)) = \sigma_i(h^t)$ whenever i is not the leader at h^t. The construction of S and the Markovian property ensure that q_i is a well-defined function. I claim that adding symmetry means that $q_i(s) = q(s)$ for all i and $s \in \mathcal{S}$. Fix any i and $\mathbf{h}^t$ such that $s = S(\mathbf{h}^t)$ and $l^{t-1} \neq i$. By construction, $q_i(s) = q_i(S(\mathbf{h}^t)) = \sigma_i(\mathbf{h}^t)$.

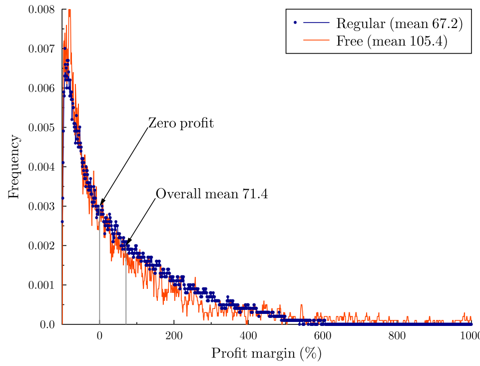


(a) Auctions with increasing prices

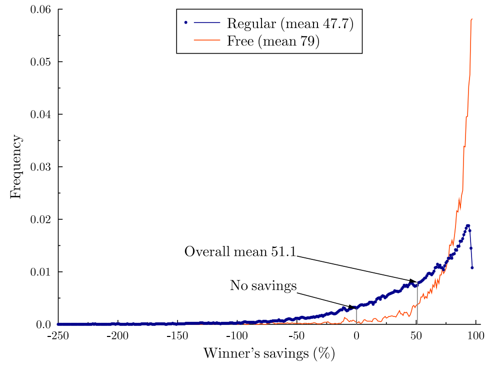


(b) Auctions with fixed prices

Figure 1: Distribution of the normalized end prices in different types of auctions



(a) Profit margin



(b) Winner's savings

Figure 2: Distribution of the profit margin and winner's savings in Regular and Free auctions.

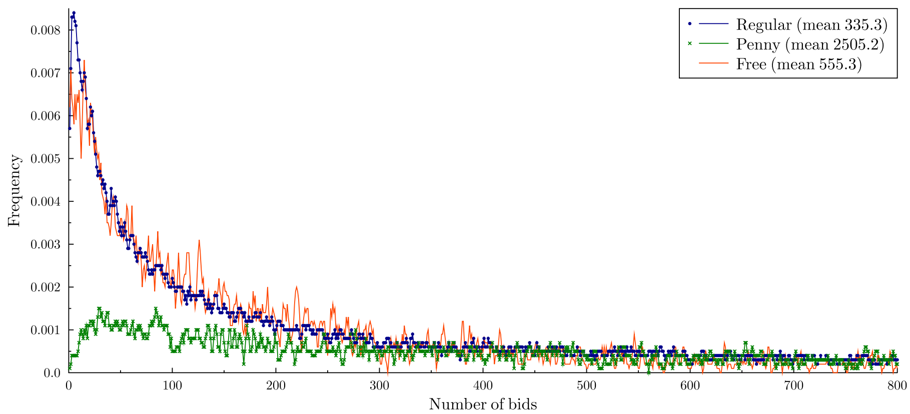


Figure 3: Distribution of the number of bids submitted in different types of auctions

Next, fix any other non-leader, $\hat{i}$, so that $l^{t-1} \neq \hat{i}$. Construct another history $\hat{\mathbf{h}}^t$ that is otherwise identical to $\mathbf{h}^t$ except that i and $\hat{i}$ are switched. Then, $S(\hat{\mathbf{h}}^t) = s$ and by symmetry, $\sigma_i(\mathbf{h}^t) = \sigma_{\hat{i}}(\hat{\mathbf{h}}^t)$. Therefore, $q_{\hat{i}}(s) = q_{\hat{i}}(S(\hat{\mathbf{h}}^t)) = \sigma_{\hat{i}}(\hat{\mathbf{h}}^t) = \sigma_i(\mathbf{h}^t) = q_i(s)$. □

This result means that if a strategy profile is symmetric and Markovian, it can be simplified. Replace σ by $\mathbf{q}$ which is just defined for all $s \in \mathcal{S}$ instead of the full set of histories $\mathcal{H}$. The following two lemmas show that (at least in the cases considered in this paper) the solution method is also simplified by these assumptions because any SMPE can be found simply by solving for stage-game Nash equilibria for each state $s \in \mathcal{S}$, taking into account the solutions to other states and the implied continuation value functions.

Lemma 3. *With $\varepsilon > 0$, a strategy profile σ is a SMPE if and only if it can be represented by $q : \mathcal{S} \to [0, 1]$ where $q(s)$ is the Nash equilibrium in the stage-game at state s, taking into account the continuation values implied by transitions S.*

Proof First, necessity: If σ is SMPE, then by Lemma 2, it can be represented by $\mathbf{q}$; additionally, because it is a SPNE, there cannot be profitable one-stage deviations.

Now, for sufficiency observe that when $\varepsilon > 0$, the price rises each period if the game has not stopped. Because rational buyers will never bid after the price rises above the value of the object, the game stops after a finite number of periods. So, although the game is (by the rules) infinite, it is equivalent in the sense of payoffs and equilibria with a game which is otherwise identical to the initial auction but where, after some finite time, the current leader gets the object at the current price. This game is finite, and checking one-stage deviations is a sufficient condition for SPNE. □

Lemma 4. *With $\varepsilon = 0$, a strategy profile σ is a SMPE if and only if it can be represented by $q : \mathcal{S} \to [0, 1]$ where $q(s)$ is the Nash equilibrium in the stage-game at state s, taking into account the continuation values implied by transitions S.*

Proof The proof for necessity is analogous to Lemma 3. For sufficiency we need to show that it suffices to verify one-shot deviations. It is useful to use short notation as follows: The probability that a non-leader submits a bid at state N is $\hat{q}$, and the expected value for a non-leader is $\hat{v}$; for the leader, the expected value is $\hat{v}^*$. Similarly, at state $N+1$ (i.e., at the initial period 0), the corresponding values are $\hat{q}_0, \hat{v}_0$, and $\hat{v}_0^*$, respectively. Finally, let the probability that the buyer who submitted a bid becomes a leader be P, which depends on $\hat{q}$ because it depends on what the opponents do, but not on the buyer's own strategy.

Consider any period after period 0. By assumption, in equilibrium, non-leaders should submit bids with probability $\hat{q}$. Clearly, $\hat{q} \in (0, 1)$ so that each of them is indifferent between bidding and not bidding. That is,

$$P\hat{v}^* + (1-P)\hat{v} - Z(P)c = \hat{v} \quad \Longleftrightarrow \quad \hat{v}^* - \hat{v} = \frac{Z(P)}{P}c, \tag{8}$$

where $Z(P) = 1$ in the simultaneous-bids case and $Z(P) = P$ in the single-bid case. Moreover, the leader's value must satisfy $\hat{v}^* = (1-\hat{q})^N v + [1-(1-\hat{q})^N]\hat{v}$ so that $\hat{v}^* - \hat{v} = (1-\hat{q})^N[v-\hat{v}]$.

Suppose that a non-leader has a profitable multi-stage deviation at period t (still assuming that opponents use strategy $\hat{q}$). Suppose at t, this strategy requires submitting a bid at point t with probability Q and assures value $V > \hat{v}$ at period t. Then, at any point in the future when the player is a non-leader again, the value must be V again because otherwise, the leader could increase the value at either point. Also, let V^* be the value for the bidder at each point in the future when the bidder is a leader. (V^* must be equal in all points for the same reason.) Then,

$$V = Q\left[PV^* + (1-P)V - Z(P)c\right] + (1-Q)V \quad \Longleftrightarrow \quad V^* - V = \frac{Z(P)}{P}c. \tag{9}$$

The game ends with probability $(1-\hat{q})^N$. Therefore, $V^* = (1-\hat{q})^N v + [1-(1-\hat{q})^N]V$ so that $V^* - V = (1-\hat{q})^N[v-V]$. As a result,

$$(1-\hat{q})^N[v-V] = \frac{Z(P)}{P} = (1-\hat{q})^N[v-\hat{v}] \quad \Rightarrow \quad V = \hat{v}. \tag{10}$$

This result contradicts the assumption that $V > \hat{v}$.

I have shown that always playing $\hat{q}$ at any period after 0 ensures the highest possible payoffs. Therefore, if there is profitable deviation at period 0, it must be one-stage deviation. But profitable deviation is not possible because it contradicts the assumption that $\hat{q}_0$ is a Nash equilibrium in the stage-game, taking into account the continuation values from $\hat{q}$ in the following periods. □

B.2 Proof of Proposition 2

Proof First, notice that there is no (symmetric) pure-strategy equilibria in this game. If $\hat{q} = 1$, the game never ends, and all players receive $-\infty$, an outcome which cannot be an equilibrium. If $\hat{q} = 0$, then $\hat{v}^* = v$, and $\hat{v} = 0$. This outcome cannot be an equilibrium because a non-leader would want to deviate and submit a bid to receive $\hat{v}^* - c$, which is more than $\hat{v}$ (by assumption, $v > c + 1 > c$). Therefore, in any equilibrium, $\hat{q} \in (0, 1)$.

In any period, a non-leader's continuation value is equal to the expected value from not submitting a bid. The other $N-1$ non-leaders submit a bid each with probability

$\hat{q}$, so the game ends with probability $(1-\hat{q})^{N-1}$ and continues from the same point with probability $1-(1-\hat{q})^{N-1}$. Therefore,

$$\hat{v} = [1-(1-\hat{q})^{N-1}]\hat{v} + (1-\hat{q})^{N-1}0 \iff \hat{v} = 0 \tag{11}$$

because $0 < \hat{q} < 1$. Consequently, the leader has $(1-\hat{q})^N$ chance to win the object; otherwise, the leader becomes a non-leader who receives $\hat{v} = 0$. Therefore,

$$\hat{v}^* = (1-\hat{q})^N v + [1-(1-\hat{q})^N]\hat{v} = (1-\hat{q})^N v. \tag{12}$$

The value of $\hat{q}$ is pinned down by the mixing condition of a non-leader,

$$\hat{v} = 0 = \sum_{K=0}^{N-1} \binom{N-1}{K} \hat{q}^K (1-\hat{q})^{N-1-K} \left[\frac{1}{K+1}\hat{v}^* + \frac{K}{K+1}\hat{v}\right] - c \iff \frac{c}{v} = (1-\hat{q})^N \Psi_N(\hat{q}). \tag{13}$$

Additionally, because $\Psi_N(q)$ is a strictly decreasing continuous function with limits 1 and $1/N$ as $\hat{q} \to 0$ and $\hat{q} \to 1$, respectively, as $\hat{q}$ changes in $(0,1)$, it takes each value in the interval $(1/N, 1)$ exactly once. The function $(1-\hat{q})^N$ is also a strictly decreasing continuous function with limits 1 and 0, so the function $(1-\hat{q})^N \Psi_N(q)$ is a strictly decreasing continuous function in $\hat{q}$ and takes all values in the interval $(0,1)$. Therefore there exists unique $\hat{q} \in (0,1)$ which solves the equation $(1-\hat{q})^N \Psi_N(\hat{q}) = c/v \in (0,1)$.

Next, I consider period 0 to find the equilibrium strategy at $\hat{q}_0$. I denote the expected value that a player gets from playing the game by $\hat{v}_0$, and I claim that $\hat{q}_0 \in (0,1)$. Suppose first that $\hat{q}_0 = 0$, a condition that would end the game instantly with all bidders getting nothing. By submitting a bid, a player could ensure becoming the leader with certainty in the next round and therefore getting value $\hat{v}^* - c = (1-\hat{q})^N v - c$. The equilibrium condition says that this value must be less than the equilibrium payoff of zero; however, in that case,

$$(1-\hat{q})^N v - c \leq 0 \iff (1-\hat{q})^N \leq \frac{c}{v} = (1-\hat{q})^N \Psi_N(\hat{q}), \tag{14}$$

so $\Psi_N(\hat{q}) \geq 1$. There is a contradiction because $\Psi_N(\hat{q}) < 1$ for all $\hat{q} > 0$.

Next, suppose that $\hat{q}_0 = 1$ is an equilibrium so that each bidder must weakly prefer bidding over not bidding and consequently getting the continuation value of a non-leader, $\hat{v} = 0$. The equilibrium condition is

$$\frac{1}{N+1}\hat{v}^* - c = \frac{1}{N+1}(1-\hat{q})^N v - c \geq 0 \iff \frac{(1-\hat{q})^N}{N+1} \geq \frac{c}{v} = (1-\hat{q})^N \Psi_N(\hat{q}), \tag{15}$$

so $\Psi_N(\hat{q}) \le 1/(N+1) < 1/N$, which is also a contradiction.

Thus, in equilibrium, $0 < \hat{q}_0 < 1$ is defined by

$$0 = \sum_{K=0}^{N} \binom{N}{K} \hat{q}_0^K (1-\hat{q}_0)^{N-K} \frac{1}{K+1} \hat{v}^* - c \quad \Longleftrightarrow \quad (1-\hat{q})^N \Psi_{N+1}(\hat{q}_0) = \frac{c}{v}. \tag{16}$$

The following equivalent form of the equation shows that the previous equation defines $\hat{q}_0$ uniquely for a fixed $\hat{q} \in (0,1)$:

$$(1-\hat{q})^N \Psi_{N+1}(\hat{q}_0) = \frac{c}{v} = (1-\hat{q})^N \Psi_N(\hat{q}) \quad \Longleftrightarrow \quad \Psi_{N+1}(\hat{q}_0) = \Psi_N(\hat{q}). \tag{17}$$

I have already shown that $\Psi_N(\hat{q}) \in (1/N, 1)$. As argued above, (continuous, strictly decreasing) $\Psi_{N+1}(\hat{q}_0)$ takes values in the interval $(1/(N+1), 1) \supset (1/N, 1)$, so the equation must have a unique solution $\hat{q}_0$.

Proof for the single-bid case is analogous for the most part with the exception that c must be paid only by the new leader, so the mixing condition for a non-leader is

$$\hat{v} = 0 = \sum_{K=0}^{N-1} q^K (1-q)^{N-1-K} \left[\frac{1}{K+1} [\hat{v}^* - c] + \frac{K}{K+1} \hat{v} \right] \Longleftrightarrow c = \hat{v}^* = (1-q)^N v \tag{18}$$

so that $\hat{v}$ is defined by the condition $(1-\hat{v})^N = c/v$. Similarly, $\hat{v}_0$ is defined by the equation $(1-\hat{v}_0)^{N+1} = c/v$, both sides of which are special cases of the expressions given in the proposition with $\Psi_K(q) = 1$.

Consequently, in period 0, buyers are always indifferent between bidding and not bidding because both give zero payoff under any realization of uncertainty. Therefore, each $\hat{q}_0 \in [0,1]$ can be an equilibrium strategy. $\square$

B.3 Proof of Corollary 1

Proof

(i) The simultaneous-bids case

(a) The proposition showed that the probability that an individual bidder submits a bid at $t > 0$ is a constant $\hat{q} \in (0,1)$; therefore, the probability that any bidder submits a bid is also constant, $\hat{Q} = 1 - (1-\hat{q})^N \in (0,1)$.

For comparative statics, note that $\Psi_N(q)$ is decreasing both in q and in N. Also, $(1-q)^N$ is decreasing both in q and in N; therefore, $(1-q)^N \Psi_N(q)$ is decreasing in both arguments. Because $\hat{q}$ is defined by $(1-q)^N \Psi_N(q) = c/v$, $\hat{q}$

is decreasing in N and c/v. Because $\hat{Q} = 1 - (1-\hat{q})^N$ is increasing in $\hat{q}$, $\hat{Q}$ is decreasing in c/v, as well.

Finally, $\hat{Q}$ is decreasing in N as shown by rewriting the definition of $\hat{q}$ in terms of $\hat{Q}$ as

$$(1-\hat{Q})\frac{\hat{Q}}{N[1-(1-\hat{Q})^{\frac{1}{N}}]} = \frac{c}{v}. \tag{19}$$

It suffices to show that the left-hand side is strictly increasing in $\hat{Q} \in (0,1)$ and strictly decreasing in $N \geq 2$.

Because the expression was decreasing in $\hat{q}$, it is also decreasing in $\hat{Q}$. To prove that it is decreasing in N, I show that the denominator, $N[1-(1-\hat{Q})^{\frac{1}{N}}]$, is increasing in N. Fix $N \geq 2$ and $\hat{Q} \in (0,1)$, and define $x = (1-\hat{Q})^{\frac{1}{N(N+1)}}$. Then $x \in (0,1)$, and I need to show that

$$(N+1)\left(1-(1-\hat{Q})^{\frac{1}{N+1}}\right) > N\left(1-x^{N+1}\right) = N\left(1-(1-\hat{Q})^{\frac{1}{N}}\right), \tag{20}$$

which can be rewritten as

$$Nx^N(1-x) < 1-x^N \iff x^N < \frac{1-x^N}{N(1-x)} = \frac{1+x+x^2+\cdots+x^{N-1}}{N}. \tag{21}$$

This equation is satisfied due to the following: for all $x \in (0,1)$ and $N \geq 2$, $x^K > x^N$ for all $K < N$. Therefore, $\hat{Q}$ is increasing in N.

(b) If $N+1 > 2$, then by the proposition, $\hat{q}_0 \in (0,1)$; therefore, $\hat{Q}_0 = 1-(1-\hat{q})^N \in (0,1)$. Because $\Psi_{N+1}(\hat{q}_0) = \Psi_N(\hat{q})$ and $\Psi_N(q)$ is decreasing both in q and in N, $\hat{q}_0$ is strictly decreasing in c/v and $\hat{q}_0 < \hat{q}$. This result also implies that $\hat{Q}_0$ is strictly decreasing in c/v.

(ii) The single-bid case

(a) This proof is analogous to the proof for the simultaneous-bids case except for the relationship between $\hat{Q}$ and N. Note that $\hat{q}$ is defined by $(1-\hat{q})^N = c/v$ so that $\hat{Q} = 1 - v/c$, which is independent of N.

(b) Any $\hat{q}_0 \in [0,1]$ can be an equilibrium strategy by the proposition. It follows that $\hat{Q}_0 = 1-(1-\hat{q}_0)^{N+1}$ can take any values in $[0,1]$, as well.

(iii) In both the simultaneous-bids case and the single-bid case, bidders are indifferent between bidding and not bidding in each period so that one strategy over which they randomize is never bidding—a strategy that gives a value of zero.

(iv) There is another way to compute the ex-ante value to players, $\hat{v}_0$. Let the actual number of bids the players submitted in a particular realization of uncertainty be B. Conditioning on sale means that $B > 0$. Let $Pr(B|B > 0)$ denote the conditional probability that the number of bids is B. Because the value to the winner is v and all the players collectively paid $B \cdot c$ in bid costs, the aggregate value to the players is $v - Bc$. Due to symmetry and risk-neutrality, this value is ex-ante divided equally among all players, so

$$0 = \hat{v}_0 = \frac{1}{N+1} \sum_{B=1}^{\infty} [v - Bc] Pr(B|B > 0) = \frac{v - cE(B|B > 0)}{N+1}. \tag{22}$$

Revenue to the seller, given that the object is sold, is Bc. Therefore, the expected revenue is $\overline{R} = E(Bc|B > 0) = cE(B|B > 0) = v$.

□

B.4 Proof of Lemma 1

Proof If $p > v$, the upper bound of the winner's payoff in this game is $v - p < 0$, so any continuation of this game is worse for all the players. Therefore, prices for which $q(p) > 0$ are bounded by v.

Let $\hat{p}$ be the highest price for which $q(\hat{p}) > 0$. Suppose by contradiction that $\hat{p} \geq \lfloor v - c \rfloor$. Because $q(\hat{p} + K) = 0$ for all $K \in \mathbb{N}$, the game ends instantly if the price rises above $\hat{p}$. Therefore, $v(\hat{p} + K) = 0$, and

$$v^*(\hat{p} + K) = v - (\hat{p} + K) \leq (v - c) - \lfloor v - c \rfloor - K + c < c, \quad \forall K \in \mathbb{N} \tag{23}$$

because by definition $(v - c) - \lfloor v - c \rfloor < 1$; thus, $(v - c) - \lfloor v - c \rfloor - K < 0$ for all $K \in \mathbb{N}$.

Therefore, submitting a bid with positive probability at $\hat{p}$ gives an expected payoff that is strictly lower than the cost of submitting the bid, c, which is a contradiction. □

B.5 Proof of Proposition 3

Proof The construction in Section 5 describes the method to find equilibrium q. The conditions C1, C2, and C3 are written so that there are no profitable one-stage deviations. Proving the existence of SMPE only requires proof that there is at least one q that satisfies at least one of three sets of conditions.

Each round in the auction is a finite, symmetric strategic game. Theorem 2 in Nash (1951) proves that each symmetric strategic game has at least one symmetric equilibrium.

Because the conditions are constructed so that any mixed- or pure-strategy–stage-game Nash equilibrium would satisfy them, there exists at least one such q.

Appendix C gives a simple example in which the equilibrium is not unique. $\square$

B.6 Proof of Corollary 3

Proof

1. The argument for $\overline{R} \leq v$ is analogous to Corollary 1. The aggregate expected value to the players must be equal to v minus the sum the of payments to the seller; because participation is voluntary, the sum of the payments cannot be larger than v.

   If $q(p) < 1$ for all p, then this mixed strategy puts strictly positive probability on the pure strategy in which the player never bids. The pure strategy gives 0 with certainty, so $v(0) = 0$ for all players, and $\overline{R} = v$ by implication.

   If $q(p) = 1$ for some p, then the previous argument does not work because the player does not put positive probability on the never-bidding, pure strategy. The example in Appendix C gives an equilibrium in which $q(0) \in (0,1), q(1) = 0$, but $q(2) = 1$, and $E(R|p > 0) = 8.62 < 9.1 = v$.

2. Starting with the case $\gamma > 0$, by Lemma 1, at $p = \lfloor v - c \rfloor = v - c - \gamma$, $q(p) = 0$; therefore, $v^*(\lfloor v - c \rfloor) = c + \gamma$, and $v(\lfloor v - c \rfloor) = 0$.

   Take $p = \lfloor v - c \rfloor - 1$. Then C2 cannot be satisfied because $v^*(p+1) - c = \gamma > 0$. Therefore, $q(\lfloor v - c \rfloor - 1) > 0$. When $\gamma = 0$, there is no contradiction with C2 at $p = \lfloor v - c \rfloor - 1$, but there is one at $p = \lfloor v - c \rfloor - 2$ by the same argument.

   Because $\bar{N}$ simultaneous bids raise the price by $\bar{N}$, conditional on sale, the game stops by some price level p^* only if there exist $\bar{N}$ sequential price levels before p, where $q(p) = 0$. I claim that the upper bound for the support of the final prices cannot be lower than $\lfloor v - c \rfloor$. Suppose, by contradiction, there exists $p < \lfloor v - c \rfloor$ such that $q(p) = q(p-1) = 0$. Then $v^*(p) = v - p > v - \lfloor v - c \rfloor \geq c$. Then, at $p - 1$, $v^*(p - 1 + 1) - c > 0$, so C2 cannot be satisfied; consequently, there is a contradiction.

   Therefore, when $\gamma = 0$, the game reaches $\lfloor v - c \rfloor - 2$ with positive probability, so the highest price reached with positive probability is $p^* \geq \lfloor v - c \rfloor - 2 + \bar{N} \geq v - c$. When $\gamma > 0$, $p^* \geq \lfloor v - c \rfloor - 1 + \bar{N} > v - c$.

3. As shown above, conditional on sale, the final price is $p^* \geq v - c$ with positive probability, so revenue must be at least

$$R \geq (c+1)(v-c) = v + (c+1)(v-c-1). \tag{24}$$

□

B.7 Proof of Corollary 4

Proof

1. If $\gamma > 0$:

   0) At $p = \lfloor v-c \rfloor = v - c - \gamma$, $v^*(p) = v - p = c + \gamma$, and $v(p) = 0$.
   1) At $p = \lfloor v-c \rfloor - 1$, $v^*(p+1) - c = \gamma > 0 = v(p+1)$, so only C1 is satisfied; thus, $q(p) = 1, v^*(p) = 0, v(p) = \gamma/\bar{N}$.
   2) At $p = \lfloor v-c \rfloor - 2$, $v^*(p+1) = 0 < v(p+1)$, so only C2 is satisfied; thus, $q(p) = 0$, and $v^*(p) = v - p = c + \gamma + 2$ and $v(p) = 0$.

   K) Repeat the same pattern: At each $p = \lfloor v-c \rfloor - K$ with odd K, only C1 can be satisfied, and with even K, only C2 can be satisfied.

   Therefore, at $p = 0$, C1 is satisfied if $\lfloor v-c \rfloor$ is odd, and C2 is satisfied if it is even, so the SMPE is unique.

2. If $\gamma = 0$, there is a continuum of equilibria including:

   (a) Proof by induction:

      0) At $p = \lfloor v-c \rfloor = v - c$, $v^*(p) = v - p = c$, and $v(p) = 0$.
      1) At $p = v - c - 1$, $v^*(p+1) - c = v(p+1) = 0$, so C3 is satisfied for any q. Therefore, q can be chosen such that $v^*(p) = (1-q)^{\bar{N}}(c+1) = c$; that is, $q(p) = \left[\frac{c}{c+1}\right]^{\frac{1}{\bar{N}}}$, and $v(p) = v(p+1) = 0$.

      K) At any $p = \lfloor v-c \rfloor - K$, $v^*(p+1) - c = v(p+1) = 0$, so C3 is satisfied for any q. Therefore, q can always be chosen such that $v(p) = (1-q)^{\bar{N}}(c+K) = c$, and $v(p) = 0$.

      This construction determines a unique sequence $q(1), q(2), \ldots,$ but not $q(0)$, which can be arbitrary. Because each $q(p)$ is interior, the probability that the game ends at any $p \in \{1, \ldots, \lfloor v-c \rfloor\}$ is strictly positive.

(b) The construction is the same as in part 1 of the proof, but the inequalities are weak when $\gamma = 0$, so the equilibrium is not unique.

$\square$

Appendix C Penny Auction with Multiple Equilibria

Let $N + 1 = 3, v = 9.1, c = 2$, and $\varepsilon > 0$, and suppose that the simultaneous-bids assumption holds. There are three SMPE, which are described in Table 4.

Table 4: Penny auction with multiple equilibria. The table describes the probability $q(p)$ with which non-leaders bid at each price level p. Note that the three equilibria differ only at price levels $p \leq 2$.

Price p	Equilibrium 1 $q(p)$	Equilibrium 2 $q(p)$	Equilibrium 3 $q(p)$
0	0.5090	0.5266	0
1	0	0	0.7473
2	1	0.7249	0
3		0.6996	
4		0	
5		0.4287	
6		0.0645	
≥ 7		0	